Closing Tue: TN 1, 2
Closing Next Thu: TN 3
Final: Sat, June 3rd, 5:00-7:50pm, KANE 130

TN 2 & 3: Higher order approx.

Recall:

1st Taylor polynomial and error bound

$$T_1(x) = f(b) + f'(b)(x - b)$$

On an interval [a,b], if $|f''(x)| \leq M$,

then $|f(x) - T_1(x)| \leq \frac{M}{2}|x - b|^2$.

From last time:

Consider $f(x) = \ln(x)$ at $b = 1$.

For the interval $J = [0.5,1.5]$, we found

step 1: $|f''(x)| = \left|-\frac{1}{x^2}\right| \leq 4 = M$

step 2:

$$|f(x) - T_1(x)| \leq \frac{4}{2}|x - 1|^2$$
$$\leq 2|1.5 - 1|^2 = 0.5$$

Entry Task: For the same function…

1. For any number *a* (with 0 < a < 1), give a formula for the error bound on the interval $[1 - a, 1 + a]$.
2. For what value of *a* is the error less than 0.01?

Example (you do):

Let $f(x) = x^{1/3}$ and $b = 8$.

Find the 1st Taylor Polynomial.

Use Taylor's inequality to give a bound on the error over the interval $J = [7,9]$.

(TN 2/ 3): Higher Order Approx.

The **2nd Taylor Polynomial** (or quadratic approximation) is given by

$$T_2(x) = f(b) + f'(b)(x-b) + \frac{1}{2}f''(b)(x-b)^2$$

The **quadratic error bound theorem** (Taylor's inequality) states:

on a given interval [a,b],

if $|f'''(x)| \leq M$, then

$$|f(x) - T_2(x)| \leq \frac{M}{6}|x-b|^3$$

Example:

Find the 2nd Taylor polynomial for $f(x) = x^{1/3}$ based at b = 8 and find the error bound on the interval J = [7,9].

Taylor Approximation Idea:
If two functions have all the same derivative values, then they are the same function (up to a constant).

To explain, let's compare derivatives of f(x) and $T_2(x)$ at b.

$$T_2(x) = f(b) + f'(b)(x-b) + \frac{1}{2}f''(b)(x-b)^2$$

$$T_2'(x) = f'(b) + \frac{1}{2}f''(b)2(x-b)$$
$$= f'(b) + f''(b)(x-b)$$

$$T_2''(x) = f''(b)$$

$$T_2'''(x) = 0$$

Now plug in x = b to each of these.
What do you see?

Questions:
Why did we need a ½ ?

What would $T_3(x)$ look like?

n^{th} Taylor polynomial

$$f(b) + f'(b)(x-b) + \frac{1}{2}f''(b)(x-b)^2 + \frac{1}{3!}f'''(b)(x-b)^3 + \cdots + \frac{1}{n!}f^{(n)}(b)(x-b)^n$$

In sigma notation:

$$T_n(x) = \sum_{k=0}^{n} \frac{1}{k!} f^{(k)}(b)(x-b)^k$$

Taylor's Inequality (error bound):
on a given interval [a,b],
if $|f^{(n+1)}(x)| \le M$, then

$$|f(x) - T_n(x)| \le \frac{M}{(n+1)!}|x-b|^{n+1}$$

Side Note:

For a fixed constant, *a*, the expression $\frac{a^k}{k!}$ goes to zero as k goes to infinity.

So the expression $\frac{1}{(n+1)!}|x-b|^{n+1}$, will always go to zero as *n* gets bigger.

Which means that the error goes to zero, unless something unusual is happening with M, which will see in examples later.

Example:

Find the 9th Taylor polynomial for $f(x) = e^x$ based at $b = 0$, and give an error bound on the interval $[-2,2]$.

$f(x) = e^x$ as well as $T_1(x)$, $T_2(x)$, $T_3(x)$, $T_4(x)$ and $T_5(x)$ are shown: